AN INTRODUCTION TO INDEX THEORY ON SINGULAR SPACES

PIERRE ALBIN

Introduction

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References for these lecture notes:
Some of the results discussed below are classical and well-known and some are work I've done with my collaborators. The discussion below of the de Rham operator and the signature of stratified spaces refers to work done in the papers [5, 6]. We also studied the relation to topologically defined cohomologies in [2], application to the Novikov conjecture in [7], and to mapping surgery to analysis in [8]. All but the last of these are covered in my survey paper [1]. The discussion below of Dirac-type operators and an index formula refers to work done in [3, 4].

These papers build on the seminal work of Cheeger [13–16]. Analysis on spaces with conical singularities dates back to Kondratev [21] and the approach below dates back to [26, 28]; we follow, e.g., [18, 22, 24, 27]. The index formula for spaces with isolated conical singularities is well-known, see e.g., [17, 23]. For non-isolated conic singularities, see e.g., [9, 12]. The general case of a stratified space is treated in [3].

1. First lecture: Conical singularities

1.1. **Domains.** When studying an elliptic operator on space with singularities, one of the first things to grapple with is the choice of domain. On a closed (or even complete) manifold every elliptic differential operator has a unique extension from smooth functions to a closed[1] operator on square-integrable functions.[2] This can fail once the manifold is not complete, and different choices will yield different indices (if they are even Fredholm).

Consider the Laplacian on the unit disk. It's well-known that every continuous function on the sphere has a harmonic extension into the disk. Thus the null space of the Laplacian on its largest possible domain will be infinite dimensional. On the other hand if we consider the Laplacian on a domain with vanishing boundary values, the maximum principle guarantees that there will not be any null space.

1.2. **$\mathcal{I}$-smooth functions and the Mellin transform.** Given a function f defined on the positive real axis $\mathbb{R}^+_s$, its Mellin transform is the function $\mathcal{M}(f)(\zeta)$ defined on the complex plane by the relation

$$\mathcal{M}(f)(\zeta) = \int_0^\infty f(s) s^\zeta \, \frac{ds}{s}.$$

[1]A closed operator between two topological vector spaces is one whose graph is closed.

[2]This is known as Gaffney's theorem.

This transform is the analogue for the multiplicative group of positive real numbers[3] of the Fourier transform, to which it is related by a simple change of variables. Typically, the Mellin transform of f is defined for complex numbers $\zeta = \xi + i\eta$ restricted to be in a strip $\{a < \xi < b\}$ (known as the fundamental strip of f), and often extends as a meromorphic function on some larger strip.

For example, the Mellin transform of e^{-s} is

$$\mathcal{M}(e^{-s})(\zeta) = \int_0^\infty e^{-s} s^\zeta \frac{ds}{s} = \Gamma(\zeta),$$

the Gamma function. Since e^{-s} is bounded and decays exponentially at infinity, it is easy to see[4] that is a holomorphic function for $\mathrm{Re}(\zeta) > 0$. On the other hand, the exponential function satisfies

$$e^{-s} \sim \sum_{k \geq 0} \frac{(-s)^k}{k!} \quad \text{as } s \to 0$$

and so, for any $N \in \mathbb{N}$ we have

$$\Gamma(\zeta) = \int_1^\infty e^{-s} s^\zeta \frac{ds}{s} + \int_0^1 \left(e^{-s} - \sum_{0 \leq k < N} \frac{(-s)^k}{k!} \right) s^\zeta \frac{ds}{s} + \int_0^1 \left(\sum_{0 \leq k < N} \frac{(-s)^k}{k!} \right) s^\zeta \frac{ds}{s}$$

The first term is holomorphic on the entire complex plane, the second term is $\mathcal{O}(s^N)$ as $s \to 0$ and hence holomorphic on the half-plane $\{\mathrm{Re}\,\zeta > -N\}$, and the last term can be explicitly integrated

$$\sum_{0 \leq k < N} \frac{(-1)^k}{k!} \frac{1}{k + \zeta}$$

and recognized as meromorphic on the entire complex plane, with simple poles at $\zeta \in \{0, -1, -2, \dots, -N+1\}$. Since N was arbitrary we see that $\Gamma(\zeta)$ is a meromorphic function on $\mathbb{C}$ with simple poles at $-\mathbb{N}_0$.

The same computation would have worked starting at any function $f \in \mathcal{C}^\infty([0, \infty))$ with exponential decay near infinity. That is, if f has Taylor expansion

$$f(s) \sim \sum_{k \geq 0} a_k s^k$$

around $s = 0$, then the Mellin transform of f, $\mathcal{M}(f)(\zeta)$, is holomorphic on $\{\mathrm{Re}\,\zeta > \min\{k : a_k \neq 0\}\}$ and extends meromorphically to the complex plane with at worst simple poles at $-\mathbb{N}_0$ where it satisfies

$$\mathcal{M}(f)(\zeta) = \frac{a_k}{k + \zeta} + \mathcal{O}(1) \text{ near } -k \in \mathbb{C}.$$

This connection between the asymptotic expansion of a function at zero[5] and the pole structure of its Mellin transform is one of the reasons why the Mellin transform is so useful.

[3] Note, for example, that the measure $\frac{ds}{s}$ is invariant under dilations $s \mapsto as$, $a > 0$.

[4] For example, by Morera's theorem we just need to check that it integrates to zero along the boundary of triangles in this region and this follows from Fubini's theorem.

[5] There is an analogous story involving asymptotic expansions at infinity. One can repeat the argument above or just note that if $\phi(s) = f(\frac{1}{s})$ then $\mathcal{M}(\phi)(\zeta) = \mathcal{M}(f)(-\zeta)$.

Importantly for us, the same thing works for more generally for "$\mathcal{I}$-smooth functions" (also known as polyhomogeneous functions). Here $\mathcal{I}$ is an *index set*, meaning

$$\mathcal{I} \subseteq \mathbb{C} \times \mathbb{N}_0 \text{ such that, for any } N \in \mathbb{N}, |\{(\nu,p) \in \mathcal{I} : \mathrm{Re}(\nu) < N\}| < \infty,$$

and we say that a smooth function f on $(0,\infty)$ is $\mathcal{I}$-smooth on $[0,\infty)$ if it satisfies[6]

$$f(s) \sim \sum_{(\nu,p)\in\mathcal{I}} a_{(\nu,p)} s^{\nu} (\log s)^p \text{ as } s \to 0$$

for some set of coefficients $a_{(\nu,p)}$ independent of s. [7] As we have, whenever $\mathrm{Re}(\nu+\zeta) > 0$,

$$\int_0^1 a_{(\nu,p)} s^{\nu+\zeta} (\log s)^p \frac{ds}{s} = a_{(\nu,p)} \left(\frac{\partial}{\partial\zeta}\right)^p \int_0^1 s^{\nu+\zeta} \frac{ds}{s} = a_{(\nu,p)} \frac{(-1)^p p!}{(\nu+\zeta)^p},$$

we see that the Mellin transform $\mathcal{M}(f)(\zeta)$ of an $\mathcal{I}$-smooth function f that decays exponentially at infinity is holomorphic on the half-plane $\{\mathrm{Re}\,\zeta > -\min\{\mathrm{Re}\,\nu : \exists(\nu,p) \in \mathcal{I}\}\}$ and extends meromorphically to the complex plane with poles at places and sizes determined by $\mathcal{I}$. In fact we see that the asymptotic expansion at $s=0$ and the singular parts of the meromorphically continued Mellin transform mutually determine each other.

Let us point out some more properties of the Mellin transform that can easily be reduced to standard properties of the Fourier transform by changing variables. First, it has an inverse given by

$$\mathcal{M}^{-1}(F)(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F(\xi + i\eta) s^{-\eta} \, d\eta;$$

specifically, if F is holomorphic on the set $\{\xi - \varepsilon < \mathrm{Re}\,\zeta < \xi + \varepsilon\}$ and satisfies $|F(\xi+i\eta)| = \mathcal{O}(|\eta|^{-2})$ then $\mathcal{M}^{-1}(F)$ is a continuous function on $(0,\infty)$ and its Mellin transform is F. Note that changing the vertical line over which we integrate will generally yield a different inverse.

Secondly, the Mellin transform is an isometry on L^2,

$$\mathcal{M} : L^2(\mathbb{R}^+; \tfrac{ds}{s}) \longrightarrow L^2(\{\xi = 0\}; d\eta),$$

and more generally,

$$\mathcal{M} : s^a L^2(\mathbb{R}^+; \tfrac{ds}{s}) \longrightarrow L^2(\{\xi = a\}; d\eta).$$

Note that $s^a L^2(\mathbb{R}^+; \tfrac{ds}{s}) \subseteq s^{a'} L^2(\mathbb{R}^+; \tfrac{ds}{s})$ whenever $a > a'$, so the image of $s^a L^2(\mathbb{R}^+; \tfrac{ds}{s})$ under the Mellin transform is

$$L^2(\{\xi = a\}; d\eta) \cap \mathrm{Hol}(\{\xi + i\eta : \xi < a\}).$$

Finally, the Mellin transform intertwines differentiation by $s\partial_s$ and multiplication by $-\zeta$,

$$\mathcal{M}((s\partial_s)f)(\zeta) = \int_0^\infty s^{\zeta} (s\partial_s) f(s) \tfrac{ds}{s} = -\zeta \int_0^\infty s^{\zeta-1} f(s) \, ds = -\zeta \mathcal{M}(f)(\zeta).$$

[6]Recall that the meaning of an asymptotic expansion like this is that for any $N, \ell \in \mathbb{N}$, $\left| f(s) - \sum_{\mathrm{Re}\,\zeta_k < N, p < \ell} a_{\nu_k} s^{\nu_k} \right|$ is bounded for s near 0 by a constant times the smallest $s^{\nu}(\log s)^p$ with $(\nu,p) \in \mathcal{I}$ not included in the sum.

[7]An index set is *smooth* if it also satisfies $(\zeta+1,p) \in \mathcal{I}$ whenever $(\zeta,p) \in \mathcal{I}$. With this condition, $\mathcal{I}$-smooth functions are a module over the smooth function on $[0,\infty)$, i.e., If f is a smooth function on $[0,\infty)$ and h is an $\mathcal{I}$-smooth function then fh is an $\mathcal{I}$-smooth function. It is also convenient to require whenever $(\zeta,p) \in \mathcal{I}$ that we also have $(\zeta,p') \in \mathcal{I}$ for all $0 \le p' \le p$.

1.3. **Conic singularities.** Let's start by thinking about the simplest singular spaces: a space $\widehat{X}$ with an isolated conic singularity at x_0. This means that, near x_0, $\widehat{X}$ looks like $C(Z) = Z \times [0,1)/(z,0) \sim (z',0)$. If we remove such a neighborhood, we end up with a smooth manifold $\widetilde{X}$ with boundary Z. The boundary comes with a map $\phi : Z \longrightarrow \{x_0\}$, which is a very simple map, but we will keep it as part of the structure as it reminds us that $\widetilde{X}$ came from $\widehat{X}$.

Let's give ourselves a simple conic metric[8] on $\widetilde{X}$, one that near the boundary has the form $dx^2 + x^2 g_Z$ for a fixed metric g_Z on Z. Here x is a 'boundary defining function', meaning a smooth non-negative function that vanishes only on $\partial\widetilde{X}$ and then to exactly first order, i.e.,

$$x \in \mathcal{C}^\infty(\widetilde{X};\mathbb{R}^+), \quad \partial\widetilde{X} = \{x=0\}, \quad dx \text{ has no zeros on } \widetilde{X}.$$

Let's study the most natural differential operator involving the Riemannian metric, the Laplacian Δ. In local coordinates we have

$$\Delta = -g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k\partial_k)$$

and a short computation near the boundary shows that this takes the form

$$-\partial_x^2 - \frac{v}{x}\partial_x + \frac{1}{x^2}\Delta_Z, \quad v = \dim Z$$

Since the coefficients are blowing-up at the boundary of $\widetilde{X}$, one natural domain for Δ consists of smooth functions that are compactly supported in the interior of $\widetilde{X}$. By duality, we could also define Δ distributionally. For many purposes, including ours, it is useful to work with Δ as an unbounded operator on square-integrable functions, $L^2(\widetilde{X})$. There are two natural domains for Δ as a closed[9] operator on L^2, one is the graph closure of the smooth functions of compact support and is known as the *minimal domain*,

$$\begin{aligned}\mathcal{D}_{\min}(\Delta) &= \text{ graph closure of } \mathcal{C}_c^\infty(\widetilde{X}^\circ) \\ &= \left\{ u \in L^2(\widetilde{X}) : \exists (u_n) \subseteq \mathcal{C}_c^\infty(\widetilde{X}^\circ) \text{ s.t. } u_n \xrightarrow{L^2} u \text{ and } (\Delta u_n) \text{ is } L^2\text{-Cauchy} \right\},\end{aligned}$$

the other is the restriction of Δ as an operator on distributions and is known as the *maximal domain*,

$$\mathcal{D}_{\max}(\Delta) = \{u \in L^2(\widetilde{X}) : \Delta u \in L^2(\widetilde{X})\}.$$

Any closed extension $(\Delta, \mathcal{D}(\Delta))$ of Δ from $\mathcal{C}_c^\infty(\widetilde{X}^\circ)$ satisfies

$$\mathcal{D}_{\min}(\Delta) \subseteq \mathcal{D}(\Delta) \subseteq \mathcal{D}_{\max}(\Delta).$$

By elliptic regularity we know that, away from the boundary, elements of $\mathcal{D}_{\min}(\Delta)$ or $\mathcal{D}_{\max}(\Delta)$ are precisely those in the second Sobolev space $H^2(\widetilde{X}^\circ)$. The issue is to understand the behavior of these elements at the boundary. Indeed we will show that a choice of domain is essentially a choice of boundary conditions, but to do that we need to find what the appropriate boundary data is. Writing

$$\Delta = x^{-2}(-(x\partial_x)^2 - (v+1)x\partial_x + \Delta_Z)$$

and recognizing the dilation invariance of $x\partial_x$ suggests the relevance of the Mellin transform.

[8]A general conic metric would be one that in a collar neighborhood has the form $dx^2 + x^2 h_x$ where h_x is a family of two tensors that restricts to a metric on each level set of x.

[9]An operator is closed if its graph is closed.

More completely, we recognize that $x^2\Delta$ is an elliptic polynomial in the vector fields $x\partial_x$, ∂_{z_i} (z_i coordinates along Z) from

$$\mathcal{V}_b = \{V \in \mathcal{C}^\infty(\widetilde{X}; T\widetilde{X}) : V \text{ is tangent to } \partial\widetilde{X}\}. \tag{1.1}$$

We say that $x^2\Delta$ is an elliptic second order b-differential operator.[10] There is a very well-developed theory of b-differential and pseudo-differential operators, for the moment we simply note some of the simpler features of this theory, analogous to well-known properties on closed manifolds. There is natural notion of b-Sobolev spaces

$$H_b^k(\widetilde{X}; \mathrm{dvol}_g) = \{u \in L^2(\widetilde{X}; \mathrm{dvol}_g) : V_1 \cdots V_k u \in L^2(\widetilde{X}; \mathrm{dvol}_g) \text{ for any } V_i \in \mathcal{V}_b\}$$

and b-differential operators define bounded operators between appropriate ones, e.g.,

$$x^2\Delta : H_b^2(\widetilde{X}; \mathrm{dvol}_g) \longrightarrow L^2(\widetilde{X}; \mathrm{dvol}_g).$$

In particular the domains of Δ satisfy

$$x^2 H_b^2(\widetilde{X}; \mathrm{dvol}_g) \subseteq \mathcal{D}_{\min}(\Delta) \subseteq \mathcal{D}_{\max}(\Delta) \subseteq H_b^2(\widetilde{X}; \mathrm{dvol}_g).$$

Significantly though, the inclusions $H_b^s(\widetilde{X}; \mathrm{dvol}_g) \hookrightarrow L^2(\widetilde{X}; \mathrm{dvol}_g)$, $s > 0$, are not compact; in addition to the improved regularity, compactness requires decay at the boundary so instead we have that

$$x^\delta H_b^s(\widetilde{X}; \mathrm{dvol}_g) \hookrightarrow L^2(\widetilde{X}; \mathrm{dvol}_g)$$

is compact for any $\delta, s > 0$.

To simplify the numerics, let us arrange to work with the measure $\frac{dx}{x}$. Notice that near the boundary we have

$$L^2([0,1)_x \times Z; \mathrm{dvol}_g) = L^2([0,1)_x \times Z; x^v\, dx\ \mathrm{dvol}_Z) = x^{-(1+v)/2} L^2([0,1)_x \times Z; \tfrac{dx}{x}\, \mathrm{dvol}_Z)$$

and, since multiplication by $x^{(1+v)/2}$ is an isometry, that studying Δ on $L^2([0,1)_x \times Z; \mathrm{dvol}_g)$ is equivalent to studying the operator $P = x^{(1+v)/2}\Delta x^{-(1+v)/2}$ on $L^2([0,1)_x \times Z; \frac{dx}{x}\, \mathrm{dvol}_Z)$,

$$\begin{array}{ccc} L^2([0,1)_x \times Z; \mathrm{dvol}_g) & \xrightarrow{\ \Delta\ } & L^2([0,1)_x \times Z; \mathrm{dvol}_g) \\ \Big\downarrow{\scriptstyle x^{(1+v)/2}} & & \Big\downarrow{\scriptstyle x^{(1+v)/2}} \\ L^2([0,1)_x \times Z;\ \frac{dx}{x}\, \mathrm{dvol}_Z) & \xrightarrow{\ P\ } & L^2([0,1)_x \times Z;\ \frac{dx}{x}\, \mathrm{dvol}_Z) \end{array}$$

The operator P is given by

$$x^{(1+v)/2}\Delta x^{-(1+v)/2} = x^{-2}(-(x\partial_x)^2 + \Delta_Z + (1+v)^2/4).$$

Multiplication by $x^{(1+v)/2}$ mediates between the closed domains of P and those of Δ.

If $u \in \mathcal{D}_{\max}(P)$, then

$$v = x^2 P u = (-(x\partial_x)^2 + \Delta_Z + (1+v)^2/4)u \in x^2 L^2([0,1)_x \times Z;\ \tfrac{dx}{x}\, \mathrm{dvol}_Z).$$

Hence, taking Mellin transform with respect to x, we have

$$\mathcal{M}(u)(\zeta) \in L^2(\{\xi = 0)\} \times Z; d\eta\ \mathrm{dvol}_Z) \cap \mathrm{Hol}(\{\xi + i\eta : \xi < 0\}),$$
$$\mathcal{M}(v)(\zeta) \in L^2(\{\xi = 2\} \times Z; d\eta\ \mathrm{dvol}_Z) \cap \mathrm{Hol}(\{\xi + i\eta : \xi < 2\})$$

[10]Here b stands for boundary. A full development of the b-calculus of pseudodifferential operators can be found in [27].

and so

$$\mathcal{M}(v)(\zeta,z) = (-\zeta^2 + \Delta_Z + (1+v)^2/4)\mathcal{M}(u)(\zeta,z) \implies \mathcal{M}(u)(\zeta,z) = (\Delta_Z - \zeta^2 + (1+v)^2/4)^{-1}\mathcal{M}(v)(\zeta,z)$$

is as a meromorphic continuation of $\mathcal{M}(u)$ to a larger half-plane. The poles of $\mathcal{M}(u)$ occur at those ζ for which $\zeta^2 \in \mathrm{Spec}(\Delta_Z + (1+v)^2/4)$, with real part in $(0,2)$. This is a finite set, say $\{\lambda_1,\ldots,\lambda_N\}$, and taking inverse Mellin transform we find that[11]

$$u = \sum_{j=1}^N u_j(z)x^{\lambda_j} + \widetilde{u}, \text{ with } \widetilde{u} \in x^{2-}L^2(\widetilde{X}; \tfrac{dx}{x}\,\mathrm{dvol}_Z) \cap \mathcal{D}_{\max}(P).$$

with u_j an eigenfunction of Δ_Z with eigenvalue λ_j.

This shows that elements of the maximal domain have asymptotic expansions. It also shows that elements of the maximal domain are always in $x^\varepsilon H^2_b(\widetilde{X}; \frac{dx}{x}\,\mathrm{dvol}_Z)$ for some fixed positive ε (indeed, any $\varepsilon < \min \lambda_j$). If u is in the minimal domain then $u_j = 0$ for all j since these coefficients are continuous with respect to the graph norm and clearly vanish for compactly supported functions. Thus the minimal domain is contained in $x^{2-}H^2_b(\widetilde{X}; \frac{dx}{x}\,\mathrm{dvol}_Z)$.

In particular note that $\mathcal{D}_{\max}(P) \hookrightarrow L^2(\widetilde{X}; \frac{dx}{x}\,\mathrm{dvol}_Z)$ is compact so $(P, \mathcal{D}_{\max}(P))$ has closed range and finite dimensional null space. The adjoint of $(P, \mathcal{D}_{\max}(P))$ is $(P, \mathcal{D}_{\min}(P))$ which also has finite dimensional null space (in fact is injective by the maximum principle), and so $(P, \mathcal{D}_{\max}(P))$ is Fredholm. It follows that all closed extensions of P (and hence of Δ) are Fredholm.

We can identify the minimal domain. So far we know that

$$x^2 H^2_b(\widetilde{X}; \tfrac{dx}{x}\,\mathrm{dvol}_Z) \subseteq \mathcal{D}_{\min}(P) \subseteq x^{2-}H^2_b(\widetilde{X}; \tfrac{dx}{x}\,\mathrm{dvol}_Z).$$

To see that

$$\mathcal{D}_{\min}(P) = \mathcal{D}_{\max}(P) \cap x^{2-}H^2_b(\widetilde{X}; \tfrac{dx}{x}\,\mathrm{dvol}_Z)$$

let $u \in \mathcal{D}_{\max}(P) \cap x^{2-}H^2_b(\widetilde{X}; \frac{dx}{x}\,\mathrm{dvol}_Z)$ so that $u_n = x^{1/n}u \in \mathcal{D}_{\min}(P)$ for all n. Let $\varepsilon > 0$ be small enough so that $\mathcal{D}_{\max}(P) \subseteq x^\varepsilon H^2_b(\widetilde{X}; \frac{dx}{x}\,\mathrm{dvol}_Z)$ and note that for any $v \in \mathcal{D}_{\max}(P)$ we have

$$\langle Pu_n, v\rangle_{L^2} = \langle x^\varepsilon P u_n, x^{-\varepsilon}v\rangle_{L^2} \to \langle x^\varepsilon Pu, x^{-\varepsilon}v\rangle_{L^2} = \langle Pu, v\rangle_{L^2}$$

and $\langle u_n, Pv\rangle_{L^2} \to \langle u, Pv\rangle_{L^2}$. It follows that u_n converges to u in the graph norm of P and hence $u \in \mathcal{D}_{\min}(P)$.

Thus elements of the minimal domain are precisely those elements of the maximal domain whose asymptotic expansion above has $u_j = 0$ for all j. We have identified

$$\mathcal{D}_{\max}(P)/\mathcal{D}_{\min}(P) \cong \bigoplus E_{\lambda_j}(\Delta_Z).$$

Moreover, it is easy to see that the for any closed domain $\mathcal{D}(P)$, the inclusion $i : \mathcal{D}_{\min}(P) \hookrightarrow \mathcal{D}(P)$ is Fredholm with index $-\dim \mathcal{D}(P)/\mathcal{D}_{\min}(P)$ (essentially by the rank-nullity theorem). It follows that

$$\mathcal{D}_{\min}(P) \subseteq \mathcal{D}(P) \subseteq \mathcal{D}_{\max}(P) \implies \mathrm{ind}(P, \mathcal{D}(P)) = \mathrm{ind}(P, \mathcal{D}_{\min}(P)) - \dim \mathcal{D}(P)/\mathcal{D}_{\min}(P).$$

[11]The notation $x^{2-}L^2$ stands for $\cap_{\varepsilon>0} x^{2-\varepsilon}L^2$. This arises because we can take inverse Mellin transform along any line $\{\mathrm{Re}\,\zeta = 2-\varepsilon\}$, but we may note be able to take inverse Mellin transform along $\{\mathrm{Re}\,\zeta = 2\}$ because $\mathcal{M}(u)$ could have poles on this line. If there are no poles on this line we may take $\widetilde{u} \in x^2L^2(\widetilde{X}; \frac{dx}{x}\,\mathrm{dvol}_Z) \cap \mathcal{D}_{\max}(\Delta)$.

It's worth emphasizing that the analysis above worked because $x^2\Delta$ is an elliptic b-differential operator. In general, a 'conic differential operator of degree k' is a differential operator D such that $x^k D$ is a b-differential operator of degree k. If this is elliptic as a b-operator then the analysis above works *mutatis mutandis*.

We should also point out that the situation was simplified by assuming that the metric had exactly the form $dx^2 + x^2 g_Z$ near the boundary. This is known as a 'product-type' conic metric and it is especially simple as it allows separation of variables near the boundary. The analysis above works for general conic metrics of the form $dx^2 + x^2 h_x$, but the computations are a bit messier. In particular the asymptotic expansion of elements in the maximal domain of a second order elliptic operator will generally have log terms[12]

For the scalar Laplacian, note that the expansion above involved eigenvalues of $\Delta_Z + (v+1)^2/4$ in the interval $(0, 2)$. If $v \geq 2$ then there are no eigenvalues in this interval and so the discussion above shows that the scalar Laplacian is *essentially self-adjoint*.

Next time we will consider $d+\delta$ we will have to examine the geometry of differential forms before we get a conic differential operator of order one, but then our tools from today will apply.

2. Second lecture: De Rham operator

2.1. **The de Rham operator on spaces with conic singularities.** Last time we were able to understand the closed extensions of the Laplacian of a conic metric by using the Mellin transform. The same procedure will work for more other operators that are natural with respect to the metric, but only once the geometry is adapted. Analysis on $\widehat{X}$ is carried out on $\widehat{X} \setminus \{p\}$ which is diffeomorphic to $\widetilde{X}$. We can also think of $\widetilde{X}$ as the space on which we can study the differential geometry of $\widehat{X}$.

As a simple example, what should be meant by a smooth function on $\widehat{X}$ given that this is not a smooth space? Let

$$\mathcal{C}^\infty_\Phi(\widetilde{X}) = \{f \in \mathcal{C}^\infty(\widetilde{X}) : i^*_{\partial\widetilde{X}} f \text{ is a constant}\}.$$

These are precisely those smooth functions on $\widetilde{X}$ that descend to continuous functions on $\widehat{X}$; they are a natural choice of smooth functions on $\widehat{X}$.

If these are our smooth functions, then the cotangent bundle should be the space spanned locally by their differentials,

$$\mathcal{V}^*_{\mathrm{w}} = \{\omega \in \mathcal{C}^\infty(\widetilde{X}; T^*\widetilde{X}) : i^*_{\partial\widetilde{X}}\omega = 0\} = \langle dx,\ x dz\rangle.$$

Using the Serre-Swan theorem, or directly, we can show that there is a vector bundle ${}^{\mathrm{w}}T^*\widetilde{X}$ which we will call the *wedge cotangent bundle*, together with a bundle map

$$j : {}^{\mathrm{w}}T^*\widetilde{X} \longrightarrow T^*\widetilde{X}$$

such that $j_*\mathcal{C}^\infty(\widetilde{X}; {}^{w}T^*\widetilde{X}) = \mathcal{V}^*_{\mathrm{w}} \subseteq \mathcal{C}^\infty(\widetilde{X}; T^*\widetilde{X})$. Informally we say that the space of sections of ${}^{\mathrm{w}}T^*\widetilde{X}$ *is* the set $\mathcal{V}^*_{\mathrm{w}}$.

Similarly, instead of studying $d + \delta$ on differential forms we should study it on the space of wedge differential forms,

$${}^{\mathrm{w}}\Omega^*(\widetilde{X}) = \mathcal{C}^\infty(\widetilde{X}; \Lambda^*({}^{\mathrm{w}}T^*\widetilde{X})).$$

[12] See the discussion in [24, §7] or [6, Lemma 3.2].

To see how this makes a difference, note that near the boundary, with respect to the natural splitting [13]

$$\Omega^q(\widetilde{X}) = \Omega^q(\partial\widetilde{X}) \oplus dx \wedge \Omega^{q-1}(\partial\widetilde{X}),$$

the operator $d+\delta$ is given by the two-by-two matrix,

$$d+\delta = \begin{pmatrix} d_Z + \frac{1}{x^2}\delta_Z & -\frac{1}{x}(v+1-2q) - \partial_x \\ \partial_x & -d_Z - \frac{1}{x^2}\delta_Z \end{pmatrix}.$$

However if we use wedge differential forms then this splitting takes the form

$${}^{\mathrm{w}}\Omega^q(\widetilde{X}) = x^q\Omega^q(\partial\widetilde{X}) \oplus dx \wedge x^{q-1}\Omega^{q-1}(\partial\widetilde{X})$$

and correspondingly we have

$$d+\delta = \begin{pmatrix} \frac{1}{x}(d_Z+\delta_Z) & -\frac{(v-q)}{x} - \partial_x \\ \frac{q}{x} + \partial_x & -\frac{1}{x}(d_Z+\delta_Z) \end{pmatrix}.$$

Let us emphasize two advantages of this expression. First the link Z only enters through its de Rham operator, $d_Z+\delta_Z$, which lends itself to inductive arguments. Secondly, if we multiply by x, this has the form

$$\begin{pmatrix} d_Z+\delta_Z & -(v-q) - x\partial_x \\ q + x\partial_x & -(d_Z+\delta_Z) \end{pmatrix}$$

which is a b-differential operator.

We may now argue exactly as we did in the first lecture and determine all possible closed extensions of $d+\delta$. What we find is that

$$\mathcal{D}_{\max}(d+\delta)/\mathcal{D}_{\min}(d+\delta) = \bigoplus_{-\frac{1}{2}<\lambda<\frac{1}{2}} E_\lambda\left(\begin{pmatrix} d_Z+\delta_Z & q-\frac{v}{2} \\ q-\frac{v}{2} & -(d_Z+\delta_Z) \end{pmatrix}\right),$$

all closed extensions are Fredholm, and given the index of one extension we know the index of them all.

In particular we see that $d+\delta$ will be essentially self-adjoint[14] if and only if there are no eigenvalues in $(-\frac{1}{2},\frac{1}{2})$. If we allow ourselves the freedom of scaling the metric g_Z on the link then we can scale away any non-zero eigenvalues in this interval. However the zero eigenvalue corresponds to harmonic differential forms on Z of degree $\frac{v}{2}$, and the dimension of this space is topological. Thus $d+\delta$ is essentially self-adjoint for some conic metric if and only if v is odd, or v is even and Z does not have cohomology in degree $\frac{v}{2}$. These spaces are called *Witt spaces.*

It is worth emphasizing that the wedge cotangent bundle is isomorphic to the usual cotangent bundle, and canonically isomorphic over the interior of $\widetilde{X}$. Thus studying $d+\delta$ as an operator on wedge differential forms is equivalent, over $\widetilde{X}^\circ$, to studying $d+\delta$ on the usual differential forms. However as we have seen the behavior at the boundary is much nicer.

On a Witt space we can choose a metric for which $d+\delta$ is essentially self-adjoint. With this domain we can talk about the index with respect to either the Gauss-Bonnet or the

[13]Recall that for any vector field V with dual one-form α, the operators $\mathfrak{e}_\alpha$ of exterior multiplication by α and $\mathfrak{i}_V$ of interior product with V satisfy $\mathrm{id} = \mathfrak{e}_\alpha\mathfrak{i}_V + \mathfrak{i}_V\mathfrak{e}_\alpha$. Applying this to $\alpha = dx$ and $V = \partial_x$ yields this decomposition near the boundary.

[14]A differential operator is essentially self-adjoint if it has a unique closed extension to an operator on L^2 and that extension is self-adjoint.

signature grading (after complexifying). The corresponding index theorems coincide with the Atiyah-Patodi-Singer index theorems for $\widetilde{X}$, as shown by Cheeger.

On a non-Witt space, we can choose a metric for which the only eigenvalue occurring above is $\lambda = 0$. Thus every element in the maximal domain of $d+\delta$ has an asymptotic expansion of the form

$$\omega = x^{-v/2}(\alpha(\omega) + dx \wedge \beta(\omega)) + \widetilde{\omega}, \quad \text{with } \widetilde{\omega} \in \mathcal{D}_{\min}(d+\delta), \quad \alpha(\omega), \beta(\omega) \in \mathcal{H}^{v/2}(Z)$$

and closed domains correspond to subspaces of $\mathcal{H}^{v/2}(Z) \oplus \mathcal{H}^{v/2}(Z)$. If we want a domain that is self-adjoint we can look at the symplectic pairing

$$\begin{aligned} \mathcal{D}_{\max}(d+\delta) \times \mathcal{D}_{\max}(d+\delta) &\longrightarrow \mathbb{R} \\ (u,v) &\longmapsto [u,v]_{d+\delta} = \langle (d+\delta)u, v\rangle_{L^2} - \langle u, (d+\delta)v\rangle_{L^2} \end{aligned}$$

known as the *boundary pairing of* $d+\delta$. Note that, directly from the definition of adjoint domain, given any closed extension $(d+\delta, \mathcal{D}(d+\delta))$ its adjoint has domain equal to the 'orthogonal complement' with respect to the boundary pairing of $\mathcal{D}(d+\delta)$. So self-adjoint domains are precisely the Lagrangian subspaces of $\mathcal{D}_{\max}(d+\delta)$ with respect to this pairing. In our simplified situation it is easy to compute the boundary pairing explicitly: Whenever ω_1, ω_2 are in $\mathcal{D}_{\max}(d+\delta)$ we have[15]

$$[\omega_1, \omega_2]_{d+\delta} = \langle \alpha(\omega_1), \beta(\omega_2)\rangle_{\mathcal{H}^{v/2}(Z)} - \langle \beta(\omega_1), \alpha(\omega_2)\rangle_{\mathcal{H}^{v/2}(Z)}.$$

Cheeger's ideal boundary conditions[16] consist of picking any subspace $V_a \subseteq \mathcal{H}^{v/2}(Z)$, and then setting

$$\mathcal{D}_{V_a}(d+\delta) = \{\omega \in \mathcal{D}_{\max}(d+\delta) : \alpha(\omega) \in V_a, \beta(\omega) \in V_a^{\perp}\}.$$

The kernel of $d+\delta$ with any such domain, restricted to form degree q, coincides with the de Rham cohomology of $\widetilde{X}$ with appropriate boundary conditions on d. The two extreme cases, $V_a = \{0\}$ and $V_a = \mathcal{H}^{v/2}(Z)$ have topological descriptions: they are intersection cohomology with lower middle perversity and upper middle perversity, respectively[17]. We refer to a choice of V_a as a *mezzoperversity* as it corresponds to a cohomology theory 'in between' the two middle perversity theories.

If we want a domain for the signature operator, we need to impose that these spaces are compatible with the Hodge star. A self-dual mezzoperversity is a choice of V_a such that

$$*V_a^{\perp} = V_a,$$

the existence of a subspace with this property is equivalent to the vanishing of the signature of Z. Any self-dual mezzo-perversity yields a Fredholm domain for the signature operator and, surprisingly, the index is independent of the choice of self-dual mezzoperversity[18].

[15] Note that $\mathcal{H}^{v/2}(Z)$ inherits an inner product from the L^2-inner product on Z (from the metric g_Z).

[16] Cheeger discussed de Rham cohomology with ideal boundary conditions in the presence of an isolated conic singularity in [13].

[17] Since our space has only conic singularities, these can be described in terms of the relative and absolute cohomology of $\widetilde{X}$. For an introduction to intersection homology see [19] and for a de Rham description see [1, §6.5].

[18] This was shown in [6] using the corresponding topological statement established by Banagl [10].

2.2. **Non-isolated conic singularities.** Let's take a look at the next simplest class of singular spaces: wedge spaces, or spaces with non-isolated conical singularities. Analysis on these spaces is in some ways very similar to that on spaces with isolated conic singularities. However there are important differences, for example there will be closed extensions of elliptic operators that are not Fredholm.

Geometrically[19], the singular points of $\widehat{X}$ form a closed manifold Y and this manifold has a 'tubular neighborhood', $\mathcal{T}_Y$ in $\widehat{X}$ that participates in a fiber bundle

$$C(Z) - \mathcal{T}_Y \longrightarrow Y$$

with fiber the cone over a closed manifold Z.

Removing one of these neighborhoods from $\widehat{X}$ we end up with a smooth manifold with boundary $\widetilde{X}$, whose boundary has a fiber bundle

$$Z - \partial\widetilde{X} \xrightarrow{\phi_Y} Y.$$

A *wedge metric* on $\widetilde{X}$ is one that near the boundary takes the form[20]

$$dx^2 + x^2 g_Z + \phi^* g_Y$$

where $g_Z + \phi^* g_Y$ is a submersion metric on $\widetilde{X}$.

Let us start by assuming that the boundary fiber bundle is trivial: $Y \times Z$. The scalar Laplacian takes the form

$$-\partial_x^2 - \tfrac{v}{x}\partial_x + \tfrac{1}{x^2}\Delta_Z + \Delta_Y$$

and multiplying by x^2 yields

$$-(x\partial_x)^2 - (v+1)x\partial_x + \Delta_Z + x^2\Delta_Y.$$

If $u \in \mathcal{D}_{\max}(\Delta)$ and $v = x^2\Delta(u)$ then[21]

$$\mathcal{M}(u)(\zeta) \in L^2(\{\xi = -\tfrac{v+1}{2})\} \times Z \times Y; d\eta \ \mathrm{dvol}_Z \, \mathrm{dvol}_Y) \cap \mathrm{Hol}(\{\xi + i\eta : \xi < -\tfrac{v+1}{2}\}),$$
$$\mathcal{M}(v)(\zeta) \in L^2(\{\xi = 2 - \tfrac{v+1}{2}\} \times Z \times Y; d\eta \ \mathrm{dvol}_Z \, \mathrm{dvol}_Y) \cap \mathrm{Hol}(\{\xi + i\eta : \xi < 2 - \tfrac{v+1}{2}\})$$

However

$$\begin{aligned} &\mathcal{M}(v)(\zeta, y, z) = (-\zeta^2 - (v+1)\zeta + \Delta_Z)\mathcal{M}(u)(\zeta, y, z) + \Delta_Y \mathcal{M}(x^2 u)(\zeta, y, z) \\ &\qquad \Longrightarrow \mathcal{M}(u)(\zeta, y, z) = (-\zeta^2 - (v+1)\zeta + \Delta_Z)^{-1}(\mathcal{M}(v)(\zeta, z) - \Delta_Y \mathcal{M}(x^2 u)(\zeta, y, z)) \end{aligned}$$

only gives us a meromorphic continuation of $\mathcal{M}(u)(\zeta, y, z)$ as a function of ζ valued in $L^2(dz, H^{-2}(dy))$. We again get a distributional expansion of u by taking inverse Mellin transform, with exponents coming from the spectrum of Δ_Z, but with a loss of regularity in the Y directions.

Explicitly, we are interested in those λ_j in the spectrum of Δ_Z such that $-\zeta^2 - (v+1)\zeta + \lambda = 0$ has a solution[22]

$$\zeta_j = -\tfrac{v+1}{2} + \mu_j, \text{ with } \operatorname{Re}\mu_j \in (0, 2).$$

[19] We are describing the structure of a Thom-Mather stratified space of depth two. See [20] for a discussion of different types of stratified space.

[20] More generally this is what the leading term behavior of a wedge metric looks like; the analysis is essentially the same for more general wedge metrics but slightly messier.

[21] We could conjugate by $x^{(v+1)/2}$ to simplify the numbers involved as we did above.

[22] Let us assume for simplicity that all solutions have multiplicity one.

An element u of the maximal domain of Δ then has an expansion of the form

$$u \sim \sum u_j(y,z) x^{-\frac{v+1}{2}+\mu_j} + \widetilde{u}$$

where $\widetilde{u} \in x^{2-}L^2(x^v\ dx\ dz; H^{-2}(dy))$. A priori the coefficients $u_j(y,z)$ have regularity L^2 in z and H^{-2} in y, however a Calderon interpolation argument[23] shows that each u_j has regularity $H^{-\mu_j}$ in y.

One instance of these distributional expansions is very well known. Suppose Z is a single point so that $\widetilde{X}$ is a manifold with boundary endowed with an incomplete Riemannian metric of the form $dx^2 + g_Y$ (i.e., there is no singularity). The Laplacian is, near the boundary, equal to $-\partial_x^2 + \Delta_Y$. Multiplying by x^2 yields

$$x^2\Delta = -x^2\partial_x + x^2\Delta_Y = -(x\partial_x)^2 + x\partial_x + x^2\Delta_Y.$$

Arguing as above we see that elements of the maximal domain of Δ have an asymptotic expansion

$$u \sim \sum u_j(y) x^{-\frac{1}{2}+\mu_j} + \widetilde{u}$$

where the $-\frac{1}{2}+\mu_j$ are zeroes of $-\zeta^2+\zeta$, i.e., $\mu_j \in \{\frac{1}{2}, \frac{3}{2}\}$. Thus we can rewrite the asymptotic expansion as

$$u \sim u_0(y) + xu_1(y) + \widetilde{u}.$$

Here $u_0(y)$, known as the Dirichlet data, has regularity $H^{-1/2}(Y)$ and $u_1(y)$, known as the Neumann data, has regularity $H^{-3/2}(Y)$. Thus we recover the usual notion of Cauchy data.

In closing, let us point out that in the setting of non-isolated conic singularities the replacement for (1.1) is the space of *edge vector fields*[24]

$$\mathcal{V}_e = \{V \in \mathcal{C}^\infty(\widetilde{X}; T\widetilde{X}) : V \text{ is tangent to the fibers of } \phi_Y : \widetilde{\partial}X \longrightarrow Y\}.$$

The associated edge Sobolev spaces are, for $k \in \mathbb{N}$, defined by

$$H_e^k(\widetilde{X}; \mathrm{dvol}_g) = \{u \in L^2(\widetilde{X}; \mathrm{dvol}_g) : V_1 \cdots V_k u \in L^2(\widetilde{X}; \mathrm{dvol}_g) \text{ for any } V_i \in \mathcal{V}_e\}.$$

The domains of Δ satisfy

$$x^2 H_e^2(\widetilde{X}; \mathrm{dvol}_g) \subseteq \mathcal{D}_{\min}(\Delta) \subseteq \mathcal{D}_{\max}(\Delta) \subseteq H_e^2(\widetilde{X}; \mathrm{dvol}_g).$$

Things are much more complicated if the boundary fiber bundle is not trivial. The Laplacians Δ_{Z_y} depend on which fiber we are on, and so does their spectrum. Making sense of an asymptotic expansion where the exponents are varying is very delicate, particularly when the exponents cross[25].

[23] See [24, §7] for details.
[24] See [24] for the theory of edge differential and pseudodifferential operators.
[25] See [22] for a discussion of this using the notion of 'trace bundle'.

3. Third lecture: Dirac operators

3.1. **Dirac operators on a wedge space.** As in the last lecture, let $\widehat{X}$ be a wedge space, i.e., a Thom-Mather stratified space of depth one, and let $\widetilde{X}$ be its resolution. Thus $\widetilde{X}$ is a manifold with boundary and its boundary has a fiber bundle

$$Z - \partial\widetilde{X} \xrightarrow{\phi_Y} Y$$

with closed manifold base and fibers. We can recover $\widehat{X}$ from $\widetilde{X}$ by collapsing the fibers of ϕ_Y.

A wedge metric on $\widetilde{X}$ is a Riemannian metric of the form $g = dx^2 + x^2 g_Z + \phi_Y^* g_Y$. The wedge cotangent bundle ${}^{\mathrm{w}}T^*\widetilde{X}$ is the bundle constructed via the Serre-Swan theorem starting from the cotangent vectors of bounded pointwise length, i.e., near the boundary locally spanned by

$$dx, \quad xdz, \quad dy.$$

It is worth emphasizing anew that xdz, which vanishes on $\partial\widetilde{X}$ as a section of $T^*\widetilde{X}$, does not vanish on $\partial\widetilde{X}$ as a section of the wedge cotangent bundle ${}^{\mathrm{w}}T^*\widetilde{X}$. Formally, we can restate this as the observation that there is no section of ${}^{\mathrm{w}}T^*\widetilde{X}$ that multiplied by x is equal to xdz. In the same vein, note that g defines a non-degenerate metric on ${}^{\mathrm{w}}T^*\widetilde{X}$.

A wedge Clifford module over $\widetilde{X}$ consists of:

(1) a complex vector bundle $E \longrightarrow \widetilde{X}$,
(2) a Hermitian bundle metric g_E,
(3) a connection ∇^E on E compatible with g_E,
(4) an action of the complexified Clifford bundle of ${}^{\mathrm{w}}T^*\widetilde{X}$,

$$cl : \mathbb{C}l({}^{\mathrm{w}}T^*\widetilde{X}, g) \longrightarrow \mathrm{End}(E)$$

compatible with the metric and connection.

This information determines a wedge Dirac-type operator

$$\eth_{\mathrm{w}} : \mathcal{C}_c^\infty(\widetilde{X}^\circ; E) \xrightarrow{\nabla^E} \mathcal{C}_c^\infty(\widetilde{X}^\circ; T^*\widetilde{X} \otimes E) \xrightarrow{cl} \mathcal{C}_c^\infty(\widetilde{X}^\circ; E),$$

where we have used that $T^*\widetilde{X}$ and ${}^{\mathrm{w}}T^*\widetilde{X}$ are canonically isomorphic over the interior of $\widetilde{X}$.

As usual, one example is the de Rham operator[26] $d + \delta$. Another example is, if $\widetilde{X}$ is spin, the spin Dirac operator.

The leading term of a wedge Dirac-type operator at $\partial\widetilde{X}$ is $x\eth|_{x=0}$ and can be identified with

$$x\eth|_{x=0} = \eth_{\partial\widetilde{X}/Y} + \tfrac{v}{2} cl(dx)$$

where $\eth_{\partial\widetilde{X}/Y}$ is a vertical family of Dirac-type operators on the fibers of ϕ_Y. We refer to $\eth_{\partial\widetilde{X}/Y}$ as the *boundary family* of $\eth$. For the de Rham operator, $d + \delta$, $\eth_{\partial\widetilde{X}/Y}$ is the family

$$y \mapsto \begin{pmatrix} d_{Z_y} + \delta_{Z_y} & \mathbf{N} - \frac{v}{2} \\ \mathbf{N} - \frac{v}{2} & -(d_{Z_y} + \delta_{Z_y}) \end{pmatrix},$$

where $\mathbf{N}$ is the vertical number operator (it multiplies a form by its vertical degree).

When $Y = \{\mathrm{pt}\}$, i.e., for isolated conic singularities, our analysis has shown that elements of the maximal domain have an asymptotic expansion with exponents determined by the

[26] After complexifying the bundle of forms, since our convention is that Dirac-type operators act on sections of complex vector bundles.

eigenvalues of $\eth_{\partial\widetilde{X}}$ in the interval $(-\frac{1}{2},\frac{1}{2})$. For the general case we run into the problem that the eigenvalues of $\eth_{\partial\widetilde{X}/Y}\big|_{Z_y}$ depend on the point $y\in Y$. We will deal with this in two ways.

The first approach works well for the de Rham operator. In this case the zero eigenspace for $\eth_{\partial\widetilde{X}/Y}$ is topological, and if we scale away all of the other eigenvalues of $(-\frac{1}{2},\frac{1}{2})$ then we may use the Mellin transform as we did before. We say that the metric g is 'suitably scaled' in this case. As before, elements of the maximal domain have an asymptotic expansion, albeit distributional,

$$u\in\mathcal{D}_{\max}(\eth_{\mathrm{dR}}) \implies u\sim x^{-v/2}(\alpha(u)+dx\wedge\beta(u))+\widetilde{u},$$
$$\text{with } \alpha(u),\beta(u)\in H^{-1/2}(Y;\mathcal{H}^{v/2}(\partial\widetilde{X}/Y)),\ \widetilde{u}\in x^{1-}H^{-1}(\widetilde{X};\Lambda^{*\mathrm{w}}T^*\widetilde{X}).$$

If $\widehat{X}$ is Witt[27], then the asymptotic expansion is just $u\sim\widetilde{u}$. It is not immediate that the maximal and minimal domains coincide, however, since $\widetilde{u}$ is in principle not in L^2. In [5], we showed that $\mathcal{D}_{\min}(\eth_{\mathrm{dR}})=\mathcal{D}_{\max}(\eth_{\mathrm{dR}})$ on Witt spaces by constructing pseudodifferential parametrices locally over points in Y.[28]

If $\widehat{X}$ is not Witt, recall that $\mathcal{H}^{v/2}(\partial\widetilde{X}/Y)$ is a flat subbundle over Y,[29] and choose $W\subseteq\mathcal{H}^{v/2}(\partial\widetilde{X}/Y)$ any flat sub-bundle. We refer to W as a mezzo-perversity and to the corresponding domain for $\eth_{\mathrm{dR}}$,

$$\mathcal{D}_W(\eth_{\mathrm{dR}})=\{u\in\mathcal{D}_{\max}(\eth_{\mathrm{dR}}):\alpha(u)\in H^{-1/2}(Y;W)\text{ and }\beta(u)\in H^{-1/2}(Y;W^{\perp})\},$$

as the Cheeger ideal boundary conditions corresponding to W. (Two obvious choice for W are the zero sub-bundle and the full bundle $\mathcal{H}^{v/2}(\widetilde{\partial}X/Y)$.) In [6], we showed that for any W this is a Fredholm domain for $\eth_{\mathrm{dR}}$ and defined a de Rham complex (i.e., a choice of domain for d yielding a Hilbert complex in the sense of [11]) whose cohomology is isomorphic to the Hodge cohomology of $(\eth_{\mathrm{dR}},\mathcal{D}_W(\eth_{\mathrm{dR}}))$.

In order for $\mathcal{D}_W(\eth_{\mathrm{dR}})$ to restrict to a domain for the signature operator (i.e., to be compatible with the grading induced by the Hodge star), we need to require $W=*W^{\perp}$. There may not be any such W, indeed a necessary condition is that the signature operators on the fibers of $\partial\widetilde{X}\xrightarrow{\phi_Y}Y$ have vanishing families index. If there is a flat bundle W satisfying this property, we say that $\widehat{X}$ is a Cheeger space. Strikingly, the index of the signature operator of a Cheeger space does not depend on the choice of W.

The second approach to choosing a domain is inspired by the analysis of the de Rham operator but works for any Dirac-type operator. Given a wedge Dirac-type operator $\eth$, the index of the boundary family $\eth_{\partial\widetilde{X}/Y}$ is the obstruction to finding a family of smoothing operators

[27]Recall that $\widehat{X}$ is Witt if v is odd dimensional or v is even dimensional but $\mathrm{H}^{v/2}(Z)=0$.

[28]The construction in [5] was extended in [6], a global parametrix construction is carried out in [3].

[29]One way to see this is to identify this bundle with the bundle of de Rham cohomology spaces, $\mathrm{H}^{v/2}_{\mathrm{dR}}(\partial\widetilde{X}/Y)$. A transition function for the fiber bundle $\partial\widetilde{X}\xrightarrow{\phi_Y}Y$ induces a transition function on this vector bundle via the pull-back of differential forms. This pull-back map is locally constant as de Rham cohomology classes are homotopy invariant, yielding a flat structure on the bundle.

$Q \in \Psi^{-\infty}(\partial\widetilde{X}/Y)$ such that $\eth_{\partial\widetilde{X}/Y} + Q$ is invertible.[30] We can extend Q from an operator defined only on the boundary to a smoothing operator $\widetilde{Q}$ acting on $\widetilde{X}$ by using Mazzeo's theory of edge differential operators[31] and then our objective is to study the operator $\eth_{\widetilde{Q}} = \eth + \widetilde{Q}$.

For $\eth_{\widetilde{Q}}$ we define the 'vertical APS domain', by

$$\mathcal{D}_{VAPS}(\eth_{\widetilde{Q}}) = \text{ graph closure of } x^{1/2}H^1_e(\widetilde{X};E) \cap \mathcal{D}_{\max}(\eth_{\widetilde{Q}}).$$

The nomenclature is justified by thinking about asymptotic expansions; we know that the domains are determined by the eigenspaces of boundary families between $(-\frac{1}{2}, \frac{1}{2})$ and taking this graph closure corresponds to taking the smallest domain where all of the contributions from the negative eigenvalues vanish. [32]

In [3,4], we show that $(\eth_{\widetilde{Q}}, \mathcal{D}_{VAPS}(\eth_{\widetilde{Q}}))$ is self-adjoint and Fredholm with compact resolvent. We construct its heat kernel and show that it has a short-time asymptotic expansion of the form

$$\mathrm{Tr}(e^{-t\eth^2_{\widetilde{Q}}}) \sim t^{-\dim \widetilde{X}/2} \sum_{j=0}^{\infty} t^{j/2}(a_j + b_j \log t).$$

We carry out the heat equation proof of the index theorem and show that, when $\widetilde{X}$ is even-dimensional and E is $\mathbb{Z}_2$-graded,

$$\mathrm{ind}(\eth_{\widetilde{Q}}) = \int_{\widetilde{X}} \widehat{A}(\widetilde{X})\,\mathrm{Ch}'(E) + \int_Y \widehat{A}(Y) \left(-\tfrac{1}{2}\widehat{\eta}(\eth_{\partial\widetilde{X}/Y} + Q) + \int_Z T\widehat{A}(\nabla^{\mathrm{con}}, \nabla^{\mathrm{cyl}}) \right)$$

where $\mathrm{Ch}'(E)$ is the 'twisted Chern character', $\widehat{\eta}$ is the Bismut-Cheeger eta form, and $T\widehat{A}(\nabla^{\mathrm{con}}, \nabla^{\mathrm{cyl}})$ is obtained by transgressing the $\widehat{A}$-genus of two connections on $Z \times \mathbb{R}^+_s$ and then restricting to $s = 0$, the two connections are the Levi-Civita connections of a conical metric and a cylindrical metric.

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[30] In fact there is more structure in the boundary family. The bundle E has an action by the full Clifford algebra of $\widetilde{X}$, while the vertical family involves only Clifford multiplication in vertical directions. It turns out [3] that $\eth_{\partial\widetilde{X}/Y}$ commutes with Clifford multiplication by dx and by $\mathbb{C}l(T^*Y)$ and so has a families index in $K(\mathbb{C}l(\langle dx\rangle \oplus T^*Y))$. If this index vanishes then we can demand that Q also commute with $\mathbb{C}l(\langle dx\rangle \oplus T^*Y)$.

[31] Introduced in [24] (see also [25] for applications to wedge boundary value problems) and extended to stratified spaces in [3].

[32] In forthcoming work with Jesse Gell-Redman, we use the theory of spectral sections from [29, 30] to show that every closed extension of $\eth$, $(\eth, \mathcal{D})$, that is Fredholm determines a perturbation Q such that $\mathrm{ind}(\eth, \mathcal{D}) = \mathrm{ind}(\eth_{\widetilde{Q}}, \mathcal{D}_{VAPS}(\eth_{\widetilde{Q}}))$.

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University of Illinois, Urbana-Champaign
E-mail address: `palbin@illinois.edu`